University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 6
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Reading:
105 Notes 7.1-7.8
Hand & Finch 4.1-4.6

1.
Show that the period of oscillation of a particle of mass m in a potential $U = A|x|^n$ is given by

$$T = \frac{2}{n}\sqrt{\frac{2\pi m}{E}}\left(\frac{E}{A}\right)^{1/n}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$$

Take $n = 2$, evaluate the gamma functions, and thus show that T reduces to the normal expression for a parabolic potential.
Solution:
$E = \frac{1}{2}mv^2 + A\,|x|^n$, so

$$v = \dot{x} = \sqrt{\frac{2}{m}}\,(E - A\,|x|^n)^{\frac{1}{2}} \ .$$

We'll just compute the time it takes the particle to go from $x = 0$ to $x = x_{\max} \equiv (E/A)^{1/n}$. This time is one fourth of the total period T. Since x is positive over this interval, we can drop the absolute value signs.

$$\begin{aligned}\tfrac{1}{4}T &= \int dt = \int_0^{x_{\max}} \frac{dt}{dx}\,dx \\ &= \sqrt{\frac{m}{2}}\int_0^{x_{\max}} \frac{dx}{\sqrt{E - Ax^n}}\end{aligned}$$

Substitute $y = Ax^n/E$, and you get

$$T = \frac{2}{n}\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{1/n}\int_0^1 \frac{y^{\frac{1}{n}-1}\,dy}{\sqrt{1-y}}$$

This integral is a beta function, which has the following properties:

$$B(p,q) \equiv \int_0^1 t^{p-1}(1-t)^{q-1}dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

In our case, $p = \frac{1}{n}$ and $q = \frac{1}{2}$. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, so

$$T = \frac{2}{n}\sqrt{\frac{2\pi m}{E}}\left(\frac{E}{A}\right)^{1/n}\frac{\Gamma(\frac{1}{n})}{\Gamma(\frac{1}{2}+\frac{1}{n})}.$$

2.
Use a Green function to obtain the response of an underdamped linear oscillator

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = F(t)$$

to a driving (acceleration) function of the form

$$\begin{aligned}F(t) &= 0 && (t < 0) \\ &= F_0 e^{-\beta t} && (t > 0)\ ,\end{aligned}$$

where γ, ω_0, F_0, and β are constants.
Solution:
The Green's function for an underdamped oscillator is

$$G(t) = \frac{1}{\omega_\gamma}e^{-\gamma t/2}\sin\omega_\gamma t.$$

So Green's method gives

$$\begin{aligned}x(t) &= \int_{-\infty}^t F(t')G(t-t')\,dt' \\ &= \frac{F_0}{\omega_\gamma}\int_0^t e^{-\gamma(t-t')/2}\sin\omega_\gamma(t-t')e^{-\beta t'}\,dt' \\ &= \frac{F_0 e^{-\beta t}}{\omega_\gamma}\int_0^t e^{(\beta-\gamma/2)u}\sin\omega_\gamma u\,du \quad (u \equiv t - t') \\ &= \frac{F_0}{\omega_\gamma\left((\beta-\gamma/2)^2+\omega_\gamma^2\right)} \\ &\times \Big[e^{-\gamma t/2}\left((\beta-\gamma/2)\sin\omega_\gamma t - \omega_\gamma\cos\omega_\gamma t\right) \\ &+ \omega_\gamma e^{-\beta t}\Big]\end{aligned}$$

3.
Consider a nonlinear damped oscillator whose motion is described by

$$\frac{d^2x}{dt^2} + \lambda \frac{dx}{dt}\left|\frac{dx}{dt}\right| + \omega_0^2 x = 0$$

The initial conditions are $x(0) = a$, $\dot{x}(0) = 0$. Use the method of perturbations to find a solution that is accurate to first order in the small quantity λ.

Solution:
We write the solution $x(t)$ as a power series in λ, and assume that, since λ is small, we can drop all terms with two or more powers of λ. Then, setting $x(t) = x_0(t) + \lambda x_1(t)$, our equation of motion becomes

$$\ddot{x}_0 + \lambda\ddot{x}_1 + \lambda x_0 \left|x_0\right| + \omega_0^2 x_0 + \lambda\omega_0^2 x_1 = 0$$

This equation must be true for all values of λ, so we really have two equations:

$$\ddot{x}_0 + \omega_0^2 x_0 = 0 \text{ and } \ddot{x}_1 + \omega_0^2 x_1 = -\dot{x}_0 \left|\dot{x}_0\right|$$

The first equation is just the simple harmonic oscillator, and the solution which meets our initial conditions is $x_0(t) = a\cos\omega_0 t$. Substituting that into the second equation, we find that the equation for x_1 looks like a driven oscillator with a funny driving force:

$$\ddot{x}_1 + \omega_0^2 x_1 = a^2\omega_0^2 \sin\omega_0 t \left|\sin\omega_0 t\right|$$

The easiest way to solve this equation is to solve it first for the first half-period $0 < \omega_0 t < \pi$ and then for the second half-period. So during the first time interval,

$$\ddot{x}_1 + \omega_0^2 x_1 = a^2\omega_0^2 \sin^2\omega_0 t = \tfrac{1}{2}a^2\omega_0^2(1 - \cos 2\omega_0 t).$$

If we guess that the particular solution is of the form $x_{1p} = A + B\cos 2\omega_0 t$, we can solve for A and B and get $x_{1p}(t) = \frac{1}{2}a^2\left(1 + \frac{1}{3}\cos 2\omega_0 t\right)$. We need to add a homogeneous solution to match the initial conditions $x_1(0) = 0$ and $\dot{x}_1(0) = 0$ (Remember: x_0 contained the initial displacement from the origin.) The homogeneous solution that does it is $x_{1h}(t) = -\frac{2}{3}a^2\cos\omega_0 t$, so the total solution is

$$x_1(t) = \frac{1}{2}a^2\left(1 + \frac{1}{3}\cos 2\omega_0 t - \frac{4}{3}\cos\omega_0 t\right)$$
$$(\text{for } 0 < \omega_0 t < \pi)\ .$$

For the second half-period, the driving force is $-a^2\omega_0^2\sin^2\omega_0 t$, exactly the negative of what it was before. So the particular solution x_{1p} will also be -1 times what it was before: $x_{1p}(t) = -\frac{1}{2}a^2\left(1 + \frac{1}{3}\cos 2\omega_0 t\right)$. But this time the initial conditions come from the fact that the position and velocity must be continuous at $t = \pi/\omega_0$. Specifically, from above equation for x_1, we have

$$x_1(\pi/\omega_0) = \tfrac{4}{3}a^2 \text{ and } \dot{x}_1(\pi/\omega_0) = 0$$

The homogeneous solution that matches these initial conditions is $x_{1h} = -2a^2\cos\omega_0 t$, so

$$x_1(t) = -a^2\left(\frac{1}{2} + \frac{1}{6}\cos 2\omega_0 t + 2\cos\omega_0 t\right)$$
$$(\text{for } \pi < \omega_0 t < 2\pi)\ .$$

In principle you'd need to repeat this process for each half-period *ad infinitum*, but we can see what's going to happen: After one full cycle the oscillator is again at rest, but it's at $x(T) = x_0(T) + \lambda x_1(T) = a(1 - \frac{8}{3}\lambda a)$, rather than at $x(T) = a$, which is where it would be if there were no damping. So it'll just repeat the same pattern as before, but with an amplitude smaller than before by this factor.

4.
Two particles moving under the influence of their mutual gravitational force describe circular orbits about one another with period τ. If they are suddenly stopped in their orbits and allowed to gravitate toward each other, show that they will collide after a time $\tau/(4\sqrt{2})$.

Solution:
It's much easier to work with the equivalent one-body problem, where r, the distance between the particles, is regarded as the distance to some fixed center of force, and the reduced mass μ takes the place of the mass. Then if R is the radius of the circular orbit, the period τ can be

found by setting the gravitational force equal to the centripetal force:

$$\frac{k}{R^2} = \frac{\mu v^2}{R} = \frac{4\pi^2 \mu R}{\tau^2}.$$

So $\tau = 2\pi\sqrt{\mu R^3/k}$.

Now let's find T, the time it takes the particles to collide, starting from rest at a distance R. By energy conservation, the speed at a distance r from the origin satisfies $\frac{1}{2}\mu v^2 - \frac{k}{r} = -\frac{k}{R}$, so

$$v = \dot{r} = -\sqrt{\frac{2k}{\mu}\left(\frac{1}{r} - \frac{1}{R}\right)},$$

where we take the negative root because r will be decreasing. Now we can get T by integrating:

$$\begin{aligned} T &= \int_R^0 \frac{dt}{dr}\,dr = \sqrt{\frac{\mu}{2k}}\int_0^R \left(\frac{1}{r} - \frac{1}{R}\right)^{-\frac{1}{2}} dr \\ &= \sqrt{\frac{\mu}{2k}}\int_0^R r^{\frac{1}{2}}\left(1 - \frac{r}{R}\right)^{-\frac{1}{2}} dr \\ &= \sqrt{\frac{\mu R^3}{2k}}\int_0^1 u^{\frac{1}{2}}(1-u)^{-\frac{1}{2}}\,du, \;\; \textit{where } u = \frac{r}{R} \end{aligned}$$

Using the definition of the Beta function from problem 1, with $p = \frac{3}{2}$ and $q = \frac{1}{2}$, we find that

$$\begin{aligned} T &= \sqrt{\frac{\mu R^3}{2k}}\,\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} \\ &= \frac{\pi}{2}\sqrt{\frac{\mu R^3}{2k}} \\ &= \frac{\tau}{4\sqrt{2}} \end{aligned}$$

5.
A spacecraft in uniform circular orbit about the sun, far from any planet, consists of a nose cone and a service module. By means of explosive bolts, the nose cone separates from the service module. The direction of motion of the nose cone is unchanged, but its orbit becomes a parabola instead of a circle. The service module falls directly into the sun. Solve for the ratio $\rho = m_{\text{cone}}/m_{\text{spacecraft}}$, where the spacecraft mass is considered to be the sum of the nose cone and service module masses.

Solution:
Let v_0 be the initial velocity of the spacecraft (before the separation). Remember that an object in a parabolic orbit has twice as much kinetic energy as an identical object in a circular orbit at the same distance. (Why? Because an object in a circular orbit has $T = -\frac{1}{2}V$, by the virial theorem, and an object in a parabolic orbit has $T = -V$, since its total energy is zero.) So during the separation, the nose cone picks up a factor of two in kinetic energy, so its speed goes up by a factor $\sqrt{2}$: $v_{\text{cone}} = v_0\sqrt{2}$. Therefore, the change in momentum of the cone is $\Delta p_{\text{cone}} = m_{\text{cone}}\Delta v_{\text{cone}} = m_{\text{cone}} v_0(\sqrt{2} - 1)$.

Now consider the service module. Its speed is reduced to zero, so its change in momentum is $\Delta p_{\text{module}} = -m_{\text{module}} v_0$. The total change in momentum is zero, so

$$\begin{aligned} m_{\text{cone}} v_0(\sqrt{2} - 1) &= m_{\text{module}} v_0 \\ m_{\text{cone}}(\sqrt{2} - 1) &= m_{\text{module}} \\ \rho = \frac{m_{\text{cone}}}{m_{\text{cone}} + m_{\text{module}}} &= \frac{1}{\sqrt{2}} \end{aligned}$$

6.
A particle moves under the influence of a central force given by $F(r) = -k/r^n$. If the particle's orbit is circular and passes through the force center, show that $n = 5$.

Solution:
The equation in polar coordinates of a circle of radius a passing through the origin is

$$r = 2a\cos\theta$$

The other two facts we'll need are conservation of energy and angular momentum. Angular momentum conservation gives

$$l = mr^2\dot{\theta}, \qquad \text{or} \qquad \dot{\theta} = \frac{l}{mr^2}$$

with l constant. We can use this to write $\dot{r}$ in terms of r. Differentiate and square the equation for the circle, and substitute for $\dot{\theta}$:

$$\begin{aligned} \dot{r}^2 &= 4a^2\sin^2\theta\,\dot{\theta}^2 \\ &= 4a^2\dot{\theta}^2(1 - \cos^2\theta) = \dot{\theta}^2(4a^2 - r^2) \\ &= \frac{l^2}{m^2 r^4}(4a^2 - r^2)\,. \end{aligned}$$

You're probably wondering why we're playing these algebra games. Well, we wanted to write the total energy of the particle without any derivatives in it. Noting that the potential energy corresponding to this force is $U(r) = -\frac{k}{(n-1)r^{n-1}}$, we can now write:

$$\begin{aligned} E &= \tfrac{1}{2}mr^2\dot{\theta}^2 + \tfrac{1}{2}m\dot{r}^2 - \frac{k}{(n-1)r^{n-1}} \\ &= \frac{l^2}{2mr^2} + \frac{l^2}{2mr^4}(4a^2 - r^2) - \frac{k}{(n-1)r^{n-1}} \\ &= \frac{2l^2a^2}{m}\frac{1}{r^4} - \frac{k}{(n-1)}\frac{1}{r^{n-1}} \end{aligned}$$

So the energy has two terms, one of which varies as r^{-4} and the other of which varies as $r^{-(n-1)}$. But E must be constant as r varies, so those two terms must cancel each other. That can only happen if their exponents are equal, so we must have $n = 5$.

7.

A spacecraft in circular orbit about the sun fires its thruster in order to change instantaneously the *direction* of its velocity $\mathbf{v}$ by 45° (toward the sun), keeping the same *magnitude* $|\mathbf{v}|$. What is the eccentricity of the spacecraft's new orbit?

Solution:

We shall use the subscript "f" to indicate immediately after the thruster fires, and "i" to indicate immediately before. We are given that

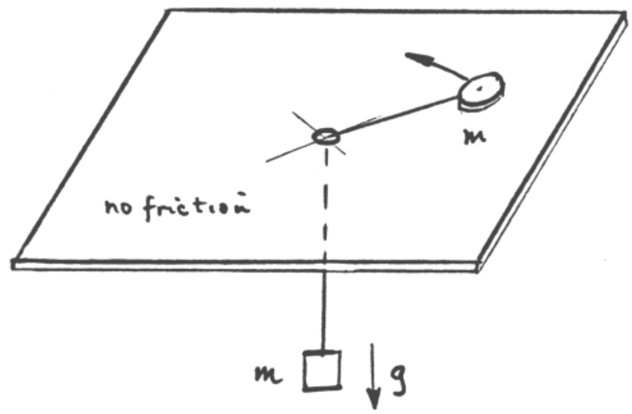
$$\vec{v}_{\mathrm{f}} = \frac{|\vec{v}_{\mathrm{i}}|}{\sqrt{2}}\left(-\hat{r} + \hat{\theta}\right) ,$$

from which we see that $l_{\mathrm{f}} = \frac{l_{\mathrm{i}}}{\sqrt{2}}$, where l is the angular momentum. Furthermore, $|\vec{v}_{\mathrm{f}}| = |\vec{v}_{\mathrm{i}}|$, and r remains instantaneously unchanged during the thrust, so $E_{\mathrm{f}} = E_{\mathrm{i}}$ as well. But $E = -\frac{k}{2a}$ (Notes 7.12), which implies that $a_{\mathrm{f}} = a_{\mathrm{i}}$. Using the definition of a from Notes 7.10 yields

$$\begin{aligned} a_{\mathrm{i}} &= a_{\mathrm{f}} \\ \frac{l_{\mathrm{i}}^2}{\mu k\,(1-\epsilon_{\mathrm{i}}^2)} &= \frac{l_{\mathrm{f}}^2}{\mu k\,(1-\epsilon_{\mathrm{f}}^2)} \\ \frac{l_{\mathrm{i}}^2}{\mu k} &= \frac{l_{\mathrm{i}}^2/2}{\mu k\,(1-\epsilon_{\mathrm{f}}^2)} \quad (\text{since } \epsilon_{\mathrm{i}} = 0) \\ 1 - \epsilon_{\mathrm{f}}^2 &= \tfrac{1}{2} \\ \epsilon_{\mathrm{f}} &= \frac{1}{\sqrt{2}} \end{aligned}$$

8.

A puck of mass m is connected by a massless string to a weight of the same mass. It moves without friction on a horizontal table, in circular orbit about the hole.



(a)

Calculate the frequency of small radial oscillations about the circular orbit.

Solution:

Let's take as our generalized coordinates r and θ, the polar coordinates of the puck with respect to the hole. Then

$$\begin{aligned} T &= T_{\text{puck}} + T_{\text{weight}} \\ &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{m}{2}\dot{r}^2 \\ &= m\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 \\ U &= mgr + \text{constant} \end{aligned}$$

and so the Lagrangian is:

$$\mathcal{L} = m\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 - mgr$$

Since θ is cyclic, angular momentum is conserved, and $l = mr^2\dot{\theta}$. Applying the Euler-Lagrange equation to the r coordinate yields

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) &= \frac{\partial\mathcal{L}}{\partial r} = mr\dot{\theta}^2 - mg \\ 2\ddot{r} &= \frac{l^2}{m^2r^3} - g \end{aligned}$$

In a circular orbit, $r = R$ and $\ddot{r} = 0$, which yields $l^2 = m^2 R^3 g$ for a circular orbit. Now, suppose we start out in a circular orbit of radius R, but then perturb it by an amount x, where $x << R$, i.e. $r = R + x$. Using the value of l for a circular orbit, the D.E. then becomes:

$$\begin{aligned}
2\ddot{x} &= g\left(\frac{R^3}{(R+x)^3} - 1\right) \\
\ddot{x} &= \frac{g}{2}\left(\frac{1}{\left(1+\frac{x}{R}\right)^3} - 1\right) \\
\ddot{x} &\approx -\frac{3g}{2R}x \;\; \text{(expand about } x = 0\text{)} \\
\omega_{\text{perturbation}} &= \sqrt{\frac{3g}{2R}}
\end{aligned}$$

(**b**)
Expressing this frequency as a ratio to the orbital frequency, show that the orbit does not close.
Solution:
From the expression for angular momentum in a circular orbit

$$\begin{aligned}
l^2 &= m^2 R^3 g = m^2 v^2 R^2 \\
v &= \sqrt{Rg}\ ,
\end{aligned}$$

and so the orbital frequency is

$$\omega_{\text{orbit}} = \frac{2\pi}{T} = 2\pi\frac{v}{2\pi R} = \sqrt{\frac{g}{R}}\ .$$

$$\frac{\omega_{\text{perturbation}}}{\omega_{\text{orbit}}} = \sqrt{\frac{3}{2}}$$

which is irrational, so the orbit does not close.